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SHORTER COMMUNICATIONS

TEMPERATURE DISTRIBUTION IN GROUND: RESPONSE FUNCTION TECHNIQUE

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NOMENCLATURE

c,	specific heat of ground [kJ/kg °C];
k,	thermal conductivity of ground [kJ/h · m °C];
$T_A(t)$,	solair temperature [°C];
$h(x,t)$,	response function [h^{-1}];
$U(x,t)$,	ground response to unit step function of $T_A(t)$ [°C];
h,	heat transfer coefficient at the surface [kJ/h · m² °C];
m,	number of harmonic;
a_0,	mean value of solair temperature [°C];
a_m,	amplitude of mth harmonic in $T_A(t)$ [°C];
t,	time [h];
x,	vertical distance below ground level [m];
s,	Laplace variable.

Greek symbols

ρ,	density of the ground [kg/m³];
ω,	2π (period)$^{-1}$ [h^{-1}];
α_0,	thermal diffusivity of ground [m²/h];
σ_m,	phase factor of mth harmonic in $T_A(t)$;
τ,	preceded time [h];
β,	a constant, $\left(\frac{h\sqrt{\alpha_0}}{k}\right)$;
γ,	a variable, $(x/\sqrt{\alpha_0})$;
μ,	a constant, $\frac{(k\omega\rho c)^{1/2}}{h\sqrt{2}}$.

INTRODUCTION

PREDICTION of temperature distribution in ground subject to arbitrary variations of solar radiation and atmospheric temperature is a problem of practical importance in evaluating the thermal flux into structures which are fully or partly underground. The existing analytical procedure[1, 2] for evaluating the temperature distribution in ground implies that the distribution is periodic in nature on account of the assumed periodicity of solair temperature which characterizes the combined effect of solar radiation and the atmospheric temperature. Such an approach is not applicable when the solair temperature is not periodic in nature; which often happens during cloudy days and other abruptly variable meteorological conditions. In fact, theoretically, the periodic solution assumes that the solair temperature is the same on all the preceding days and this is never true in actual practice. In this paper we apply the concept of response function[3, 4] to investigate the temperature distribution in ground for any arbitrary time (t) dependence of solair temperature $T_A(t)$.

THE RESPONSE FUNCTION METHOD

The essential element of response function method is the fact that the response of a constant parameter linear system to a dynamic input is given by the convolution integral as a weighted linear sum over the entire past history of the input. Thus, if the solair temperature variation at the ground surface ($x = 0$) is defined by an arbitrary function $T_A(t)$, then the temperature of the ground may be computed by the convolution integral:

$$T(x,t) = \int_0^{\infty} T_A(t-\tau)h(x,\tau)\mathrm{d}\tau \tag{1}$$

where the $h(x,\tau)$ is the ground response to a unit impulse input (or Dirac delta function) of $T_A(t)$, and hence is called the 'response function'. Further the ground response $U(x,t)$ to a unit step function of solair temperature [$T_A(t) = 0$ at $t < 0$; $T_A(t) = 1$ at $t \geq 0$] is related to the response function as below

$$h(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}U(x,t). \tag{2}$$

Ground response to a unit step function of $T_A(t)$

The ground response $U(x,t)$ is characterized by the heat conductivity equation

$$\frac{\partial}{\partial t}U(x,t) = \frac{k}{\rho c}\frac{\partial^2}{\partial x^2}U(x,t) = \alpha_0\frac{\partial^2}{\partial x^2}U(x,t). \tag{3}$$

The energy balance at the ground surface as expressed in [2] is

$$-k\frac{\partial U}{\partial x}(x,t)\bigg|_{x=0} = h[T_A(t) - U(0,t)]. \tag{4}$$

Also,

$$\left.\begin{array}{ll} T_A(t) = 0 & \text{for } t < 0 \\ T_A(t) = 1 & \text{for } t \geqslant 0 \end{array}\right\} \text{ unit step function.} \tag{5}$$

Other boundary and initial conditions are as

$$x \to \infty \quad U(x,t) \text{ is finite,} \tag{6}$$

and

$$U(x,0) = 0. \tag{7}$$

The function $U(x,t)$ may be obtained by solving equation (3) under the boundary conditions (4)–(7). This is accomplished by taking the Laplace transform of equations (3)–(7), solving them we obtain:

$$\bar{U}(x,s) = \frac{\beta}{s(\beta + \sqrt{s})}e^{-\gamma\sqrt{s}} \tag{8}$$

where

$$\beta = h\sqrt{\alpha_0}/k, \quad \text{and} \quad \gamma = x/\sqrt{\alpha_0}. \tag{9}$$

Taking the inverse Laplace transform of (8) we get

$$U(x,t) = [\mathrm{erfc}(\tfrac{1}{2}\gamma t^{-1/2}) - e^{\gamma\beta+\beta^2 t}\,\mathrm{erfc}(\tfrac{1}{2}\gamma t^{-1/2} + \beta t^{1/2})] \quad (10)$$

From equations (2) and (10), the response function may be written as

$$h(x,t) = \frac{2}{\sqrt{\pi}}\left[e^{-\gamma^2/4t}\tfrac{1}{2}\beta t^{-1/2} - \beta^2\frac{\sqrt{\pi}}{2}e^{\gamma\beta+\beta^2 t}\,\mathrm{erfc}(\tfrac{1}{2}\gamma t^{-1/2} + \beta t^{1/2})\right]; \quad (11)$$

obviously $h(x,t)$ depends only on the thermophysical properties of the ground and is independent of the nature of $T_A(t)$.

The ground temperature is given by equation (1) with response function $h(x,t)$ given by equation (11).

Ground temperature for periodic $T_A(t)$

Let

$$T_A(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\omega t - \sigma_m). \quad (12)$$

Thus, from equations (1) and (12),

$$T(x,t) = \int_0^{\infty}\left\{a_0 + \sum_{m=1}^{\infty} a_m \cos[m\omega(t-\tau) - \sigma_m]\right\} \cdot h(x,\tau)\cdot d\tau. \quad (13)$$

Using equations (11) and (13) and substituting $b_1 = (m\omega t - \sigma_m)$, $b_0 = m\omega$, $\alpha_1 = \tan^{-1} b_0/\beta^2$ and evaluating the integral we get

$$T(x,t) = a_0 + \sum_{m=1}^{\infty} I_m,$$

where

$$I_m = A(\beta/b_0^{1/2})\cos[b_1 - \pi/4 - (a\sqrt{b_0}/\sqrt{2})] - A(\beta/b_0^{1/2})\cos[b_1 + \alpha_1 - \pi/4 - (a\sqrt{b_0}/\sqrt{2})]\cos\alpha_1 + A\cos\alpha_1\cos[b_1 + \alpha_1 - (a\sqrt{b_0}/\sqrt{2})] \quad (14)$$

and

$$A = a_m e^{-\gamma\sqrt{b_0}/\sqrt{2}}. \quad (15)$$

On further simplification we get

$$T(x,t) = a_0 + \sum_{m=1}^{\infty} B_m \times \exp(-m^{1/2}\alpha x)\cos(m\omega t - \sigma_m - m^{1/2}\alpha x - \beta m) \quad (16)$$

where

$$B_m = [(1 + m^{1/2}\mu)^2 + m\mu^2]^{-1/2} a_m,$$

$$\mu = \frac{(k\omega\rho c)^{1/2}}{h\sqrt{2}} = \frac{b_0^{1/2}}{\beta\sqrt{2m}},$$

$$\alpha = \left(\frac{\omega\rho c}{2k}\right)^{1/2}$$

and

$$\beta_m = \tan^{-1}[(m^{1/2}\mu)/(1 + m^{1/2}\mu)].$$

This expression is the same as that obtained by Khatry *et al.*[2] based on periodic analysis. It establishes the equivalence of the two methods for a periodic input.

The preceding significant time in response function method

The integral in equation (1) may not always be expressible in a closed form, particularly when $T_A(t)$ is not an analytical function of time. In such cases one has to evaluate the integral by numerical computations, which in turn needs the replacement of the upper time limit infinity in the integral of equation (1) by a finite time limit (t_0) without affecting the accuracy of the results to a significant extent. The equivalence of the periodic and response function methods has been utilized to determine accuracy as a function of t_0 as follows. Fourier analysis data for daily variation of solair temperature and the other thermophysical data of ground in Kuwait as given in[2] were used to calculate the daily variation of ground temperature at $x = 0.0322$ m using the periodic analysis [equation (16)] as well as the response function method [equations (1) and (11)]. In the calculations by latter method the solair

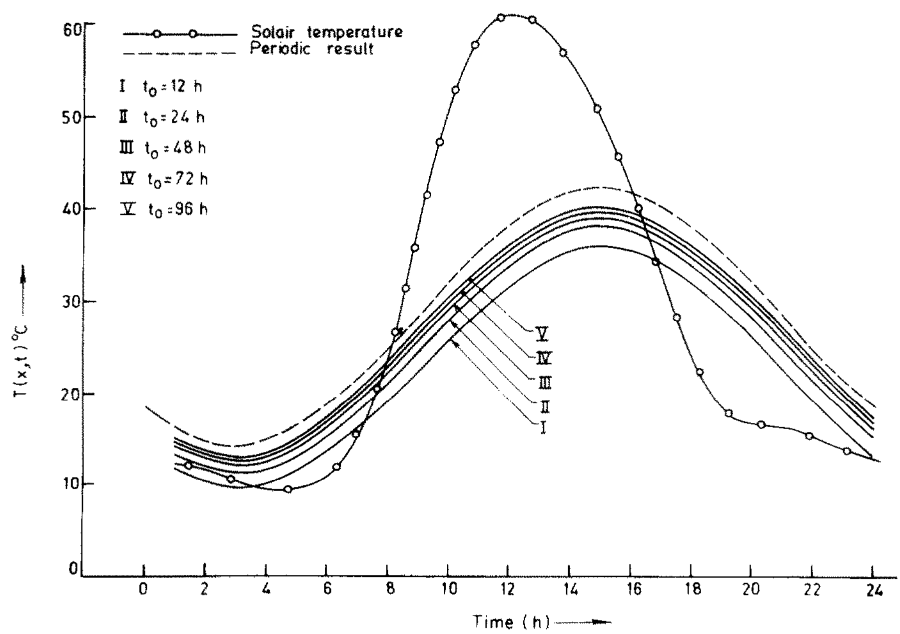

FIG. 1. Hourly variation of ground temperature with time at a depth $x = 0.0322$ m when the day under consideration is preceded by days of identical solair temperatures.

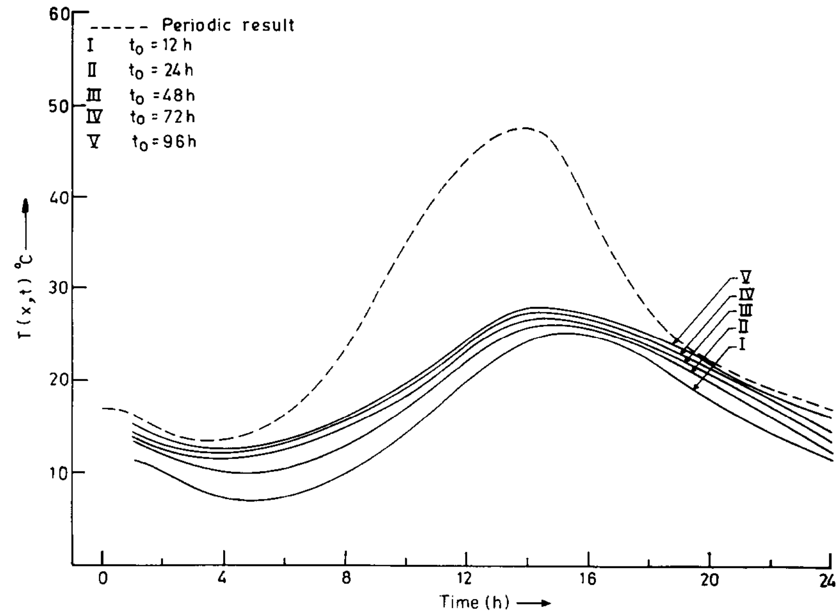

FIG. 2. Hourly variation of ground temperature with time at a depth $x = 0.0322$ m when the day under consideration is cloudy and is preceded by cloudy days of non-identical solair temperature.

temperature was assumed to be same on all preceding days and $\frac{1}{2}$, 1, 2, 3 and 4 days were respectively taken as the upper limit (t_0) in the integral of equation (1). The results of the calculations are shown in Fig. 1. It is seen that curve obtained by the response function method converges to the one obtained by harmonic analysis as t_0 increases, and a good convergence (deviation 5%) is obtained at $t_0 = 4$ days. Furthermore, in Fig. 2 we present the results of similar analysis for the case when the day under consideration as well as the preceding days are cloudy days (having their solair temperatures as given in Table 1). It is clearly seen that the curve for daily variation of ground temperature obtained by response function method does not systematically converge to the curve obtained by periodic analysis, thereby indicating the importance of response function technique for the transient, pulsed, intermittent and abruptly varying inputs.

Table 1

Time (h)	Solair temperature (°C) Day under consideration (cloudy day)	Preceding days (cloudy days) first day	second day	third day	fourth day
01	11.0	11.0	15.0	16.0	15.0
02	12.0	12.0	14.0	13.0	13.0
03	09.5	09.5	13.0	10.5	12.5
04	12.5	12.5	12.0	09.5	11.7
05	13.5	13.5	12.5	10.0	12.3
06	19.0	19.0	13.0	12.0	14.5
07	22.5	22.5	15.0	15.0	19.0
08	25.5	25.5	20.0	19.5	22.5
09	38.5	38.5	26.5	23.0	17.5
10	43.0	43.0	45.0	26.0	36.5
11	53.0	53.0	36.5	27.5	42.0
12	54.0	54.0	32.0	29.0	44.0
13	52.0	52.0	46.5	28.5	36.0
14	47.0	47.0	55.0	28.0	35.5
15	42.5	42.5	45.0	27.0	39.0
16	38.2	38.2	25.0	24.0	40.5
17	27.0	27.0	27.0	21.0	23.0
18	20.5	20.5	20.0	15.5	20.5
19	18.5	18.5	15.0	13.0	17.0
20	17.0	17.0	15.5	14.0	13.5
21	16.5	16.5	16.0	15.0	15.5
22	15.5	15.5	17.0	17.0	15.0
23	14.5	14.5	16.0	19.0	14.7
24	13.2	13.2	15.0	18.5	15.2

CONCLUSIONS

In this paper the ground temperature $T(x,t)$ due to arbitrary solair temperature is expressed in terms of response function and the convolution integral. Analytic solution of the integral is obtained for periodic variation of $T_A(t)$. The resulting expression for $T(x,t)$ is found to be same as that obtained by periodic analysis by earlier authors. The equivalence of two methods for a periodic input is used to determine the preceding significant time in response function method. Subsequently the results of response function analysis and the periodic analysis are compared for a cloudy day preceded by cloudy days.

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BEHAVIOUR OF THE TURBULENT PRANDTL NUMBER NEAR THE WALL

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NOMENCLATURE

Pr,	molecular Prandtl number $\equiv \nu/\gamma$;
Pr_t,	turbulent Prandtl number $\equiv (\overline{uv}\,\partial T/\partial y)/(\overline{v\theta}\,\partial U/\partial y)$;
Q_w,	thermometric wall heat flux;
R_{uv},	correlation coefficient $\overline{uv}/(\overline{u^2}^{1/2}\overline{v^2}^{1/2})$;
$R_{v\theta}$,	correlation coefficient $\overline{v\theta}/(\overline{v^2}^{1/2}\overline{\theta^2}^{1/2})$;
T,	local mean temperature;
T_w,	wall temperature;
T_τ,	friction temperature Q_w/U_τ;
T^+,	$(T_w - T)/T_\tau$;
U, V,	mean velocities in x, y directions, respectively;
U_τ,	friction velocity $= \tau_w^{1/2}$;
U^+,	ratio U/U_τ;
u, v, w,	velocity fluctuations in x, y, z directions, respectively;
u^+, v^+, w^+,	u/U_τ, v/U_τ and w/U_τ, respectively;
$-\overline{uv}$,	Reynolds shear stress;
$\overline{v\theta}$,	turbulent heat flux;
x, y, z,	space co-ordinates in streamwise, normal and spanwise directions;
y^+,	non-dimensional normal co-ordinate yU_τ/ν.

Greek symbols

α_i, β_i,	coefficients in equations (1)–(4);
γ,	thermal diffusivity;
δ_i,	coefficients in equations (11)–(13);
θ,	temperature fluctuation;
θ^+,	θ/T_τ;
τ_w,	kinematic wall shear stress;
ν,	kinematic viscosity.

Subscript

w,	denotes wall value.

INTRODUCTION

THE TREND of Pr_t in the region $0 < y^+ < 40$ and its possible dependence on Pr have not yet been established. Launder [1] suggested that the most sensible requirement is that any proposal of Pr_t in this region should lead to adequate predictions of measured mean temperature profiles and surface heat flux. In this context, Cebeci's [2] model indicates that, close to the wall, Pr_t increases as the wall is approached and remains constant within the viscous sublayer. The constant, as determined by Na and Habib [3] is approximately 1.43. Wassel and Catton [4] use a similar model for their calculation method, except that the constant is about 1.32, again for air. Sleicher [5] calculated Pr_t from measured velocity and temperature profiles of air in fully developed pipe flow and found that Pr_t approached a constant of about 1.4 very near the wall. This value is slightly higher than the value of $Pr^{-1/2}$ suggested, for example, by Sherwood *et al.* [6]. That Pr_t is constant, for a given Pr, very near the wall is verified by analytical considerations of mean velocity, mean temperature, Reynolds shear stress and mean heat-flux profiles in the region close to the wall. Considerations of this type have been given by Meroney [7] and Orlando *et al.* [8]. Meroney did not attempt to estimate the constant, but Orlando *et al.* suggested an experimental procedure that yields a value of about 1.4 for this constant. Although the actual value of Pr_t at the wall is not relevant to methods of calculating the heat transfer in a boundary layer, an accurate description of Pr_t in the buffer zone (approximately $5 < y^+ < 20$) can serve as a useful input to calculation methods. In the present note, the analysis followed in [7] and [8] is used with a view to establish the trend of Pr_t near the wall. This analysis is consistent with the Navier–Stokes and heat-transfer equations and yields a distribution of Pr_t, using available experimental mean velocity, temperature, momentum and heat flux profiles close to the wall.